A GENERAL DEMONSTRATION *of the* PROPERTY *of the* CIRCLE *difcovered by* MR. COTES *deduced from the* CIRCLE *only.* *By the Rev.* J. BRINKLEY, *A. M. ANDREWS' Profeffor of Aftronomy, and M. R. I. A.*

THE very elegant property of the circle difcovered by the celebrated Cotes has for its extenfive ufes always been juftly efteemed among mathematicians. The inventor left no demonftration of it; and although it immediately excited the attention of the moft eminent cultivators of the fcience, yet no general inveftigation has been hitherto given, if we except one derived from the hyperbola and impoffible expreffions, which was firft given by De Moivre, afterwards by Maclaurin and other authors. But the elegance of the theorem and the ftrictnefs of mathematical reafoning feem to require a very different kind of demonftration. The author of "Epiftola ad Amicum de inventis Cotefii," has indeed attempted a demonftration from the circle only; however it will readily appear on examination that it is not general, even conceding the demonftration

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ſtration of the theorem for expreſſing the coſine of a multiple arc in terms of the coſine of the ſimple arc. No author before Dr. Waring has given a general demonſtration of this latter theorem, and conſequently all demonſtrations of Cotes's property by the circle alone previous to his, cannot be general ſo far as that theorem is concerned, and it will be found that in another circumſtance not leſs important they are all defective. Dr. Waring in his letter to Dr. Powell has from his theorem for the chords of the ſupplement of a multiple arc ſhewn the truth of Cotes's property in particular inſtances, and in his "Propr. Algebr. Curv. Prob. 32," has given the heads of a general ſolution. But it appears one of the ſteps there omitted is the only difficult part of the demonſtration after conceding the theorem for the coſine of a multiple arc.

The demonſtration here given is general and probably as direct and ſimple as the propoſition will admit of. The proof of the lemma which it was neceſſary to premiſe is much the moſt difficult part of the whole, and it is in that ſtep of the demonſtration where the Lemma is applied that all demonſtrations heretofore have been defective and only applicable to particular inſtances.

Lemma.

Lemma.

If m and n reprefent any affirmative whole numbers: then

$$\left.\begin{array}{l}
+ : \overline{n-1}.\overline{n-2}.\overline{n-3}. \;-\;-\;-\;-\; \overline{n-m-1} \times 1 \\
- : \overline{n-2}.\overline{n-3}.\overline{n-4} \;-\;-\;-\;-\; \overline{n-m} \times m \\
+ : \overline{n-3}.\overline{n-4}.\overline{n-5}. \;-\;-\;-\;-\; \overline{n-m+1} \times m.\frac{m-1}{2} \\
-\;-\;-\;-\;-\;-\;-\;-\;- \\
-\;-\;-\;-\;-\;-\;- \\
-\;-\;-\;-\;-\;- \\
\pm : \overline{n-m+1}.\overline{n-m+2} \;-\;-\;-\; \overline{n-2m-1} \times 1
\end{array}\right\} = 0$$

Where 1, m, $m.\frac{m-1}{2}$, &c. are formed by the law of the coefficients of a binomial raifed to the m^{th}. power. The number of terms $= m+1$.

Demonstration. Let the terms of the annexed table reprefent the different expreffions for the above quantity, according to the different values of m and n.

TABLE.

Values of m	Values of n							
	1	2	3	..	..	$n-2$	$n-1$	n
2	A	B	C	..	..	..	..	..
3	A′	B′	C′	..	..	..	..	..
.	..	..	..	..	..	..	..	..
.	..	..	..	..	..	..	..	..
$m-1$	..	..	..	..	..	H	K	
m.	..	..	..	..	..	H′	K′	L.

Then

THEN 1. By ſubſtituting $n-1$ inſtead of n in the above expreſſion

$$\text{we have } K' = \begin{cases} + \overline{n-2}.\ \overline{n-3} \quad - \ - \ - \quad \overline{n-m} \times 1 \\ - \overline{n-3}.\ \overline{n-4} \quad - \ - \quad \overline{n-m+1} \times m \\ + \overline{n-4}.\ \overline{n-5} \quad - \ - \ - \quad \overline{n-m+2} \times \frac{m.\ \overline{m-1}}{2} \\ \quad \&c. \ - \ \&c. \end{cases}$$

therefore

$$L'-K' = \overline{m-1} \times \begin{cases} + \overline{n-2}.\ \overline{n-3} \quad - \ - \quad \times 1 \\ - \overline{n-3}\ \overline{n-4} \quad - \ - \quad \times m \\ + \overline{n-4}.\ \overline{n-5} \quad - \ - \quad \times m.\ \frac{\overline{m-1}}{2} \\ \quad \&c. \qquad \&c. \end{cases}$$

But by ſubſtituting $n-2$ and $n-1$ reſpectively inſtead of n, and $m-1$ inſtead of m we have

$$H = \begin{cases} + \overline{n-3}.\ \overline{n-4} \ - \times 1 \\ - \overline{n-4}.\ \overline{n-5} \ - \times m-1 \\ \quad \&c. \qquad \&c. \end{cases} \text{ and } K = \begin{cases} + \overline{n-2}.\ \overline{n-3} \ - \times 1 \\ - \overline{n-3}.\ \overline{n-4} \ - \times m-1 \\ \quad \&c. \quad \&c. \end{cases}$$

or $\overline{H+K} \times \overline{m-1} = L'-K'$ or $L' = \overline{m-1} \times \overline{H+K} + K'$.

2. Taking $m=n$ the expreſſion becomes

$$\left.\begin{array}{llll} + \overline{n-1}.\ \overline{n-2} & - \ - \ - \ - \ - & 2 \times 1 \times : 1 \\ - \overline{n-2}.\ \overline{n-3} & - \ - \ - \ - \ - & 1 \times 0 \times : m \\ - \quad - \quad - & - \ - \ - \ - & \\ - \quad - \quad - & - \ - \ - \ - & - \quad - \\ \pm \quad 0. \quad 1 & - \ - \ - & -\overline{n-3}.-\overline{n-2} \times : m \\ \mp -1.-2 & - \ - \ - & -\overline{n-2}.-\overline{n-1} \times : 1 \end{array}\right\}$$

WHICH

WHICH will be $= 0$, becaufe the firft and laft terms are the fame with contrary figns, and becaufe 0 will be a factor in each of the other terms. That the firft and laft terms will have contrary figns appears from confidering that in the laft term there are $n-1$ negative factors, and confequently when n is even the product will be negative and the fign of the term itfelf will be pofitive becaufe $m+1$ $(n+1)$ is odd, and when n is odd the product will be pofitive and the fign of the term negative.

3. SUBSTITUTING for m, 2, the general term of the firft horizontal rank $= \left.\begin{array}{l} +\overline{n-1} \\ -\overline{n-2}.\,2 \\ +\overline{n-3} \end{array}\right\} = 0.$

FROM thefe different conclufions we collect: 1ft, that (becaufe $'L = \overline{m-1}.\ H + K + K'$) if each of the terms in any horizontal rank $= 0$ the terms in the rank below are equal: 2dly, therefore it follows becaufe a term in each rank $= 0$ (when $m = n$) that if each of the terms in any horizontal rank are equal to 0, that the terms of the rank beneath are each $= 0$, and 3dly, becaufe thofe of the firft horizontal rank are each $= 0$, it follows therefore that each term of the table $= 0$. Q. E. D.

THEOREM.

1. LET the circumference of a circle be divided into n equal parts OO', $O'O''$, &c. and from a point P in the radius OC or

the radius produced without the circle draw PO, PO′, &c. then $\overline{PC}^n - \overline{OC}^n = PO \times PO' \times PO'' \times$ &c. when P is without the circle and $\overline{OC}^n - \overline{PC}^n = PO \times PO' \times PO'' \times$ &c. when P is within the circle.

2. Let the circumference be divided into $2n$ equal parts OS, SO′, O′S′, &c. then $\overline{PC}^n + \overline{OC}^n = PS \times PS' \times$ &c.

Demonstration.

1. Let OC be unity, $PC = x$; a, a', a'', &c. the cosines of o, OO′, OO″, &c.

Then will $PO^2 = x^2 + 1 - 2ax$

$PO'^2 = x^2 + 1 - 2a'x$

&c. &c.

or $PO^2 \times PO'^2 \times$ &c. $= \overline{x^2 + 1 - 2ax} \times \overline{x^2 + 1 - 2a'x} \times$ &c. $=$

$$\overline{x^2+1}|^n - \left.\begin{matrix} a \\ a' \\ a'' \\ \&c. \end{matrix}\right\} 2x.\overline{x^2+1}|^{n-1} + \left.\begin{matrix} aa' \\ aa'' \\ \&c. \end{matrix}\right\} 2^2x^2.\overline{x^2+1}|^{n-2} \quad - \quad -$$

$$\pm\, 2^n aa'a'' \,\&c. \times x^n.$$

Now if c be the cosine of any arc, the cosine of n times that arc will be $2^{n-1}c^n - n.2^{n-3}c^{n-2} + \frac{n.\overline{n-3}}{1.2}2^{n-5}c^{n-4} -$ &c. continued by successively diminishing the index of c by 2 until it becomes o or 1, and

and affixing to c^{n-u} the coefficient

$$\pm\frac{n.\overline{n-\frac{u}{2}-1}.\overline{n-\frac{u}{2}-2}\times\text{ \&c. to }\frac{u}{2}\text{ terms}}{1.\quad 2.\quad 3\ -\ -\ \frac{u}{2}}\,2^{n-u-1},$$

$+$ when $\frac{u}{2}$ is even and $-$ when odd. Hence becauſe unity is the coſine of o, P (Periphery), 2 P, 3 P, &c. it follows that if $2^{n-1}c^{n}-n.2^{n-3}c^{n-2}+$ &c. $=1$ the different values of c will be a, a', a'', &c. the coſines of o, $\frac{P}{n}$, $\frac{2P}{n}$, &c. or that the roots of the equation

$$c^{n}-\frac{nc^{n-2}}{2^{2}}+\frac{n.\overline{n-3}\,c^{n-4}}{1.\ 2.\ 2^{4}}-\ -\ -\frac{1}{2^{n-1}}=0$$ will be a, a', a'', &c.

Therefore by the nature of equations

$$a+a'+a''+\text{ \&c.}=0$$

$$aa'+aa''+\text{ \&c.}=-\frac{n}{2^{2}}$$

$$aa'a''+aa'a''+\text{ \&c.}=0$$

$$aa'a''a'''+\text{ \&c.}=+\frac{n.\overline{n-3}}{1.\ 2.\ 2^{4}}$$

&c. &c.

or generally the ſum of the products of u values a, a', a'', &c.

u being even $=\pm\dfrac{n.\overline{n-\frac{u}{2}-1}.\overline{n-\frac{u}{2}-2}.\ -\ (\text{to }\frac{u}{2}\text{ terms}}{1.\ 2.\ 3\ -\ -\ \frac{u}{2}.\ 2^{u}}$ $+$ when

$\frac{u}{2}$

$\frac{n}{2}$ is odd and — when even: alſo the product of all the values when n is odd $= \frac{1}{2^{n-1}}$ and when even $= \pm$

$$\frac{n.\overline{\frac{n}{2}-1}.\overline{\frac{n}{2}-2} \;\text{-}\;\text{-}\; (\text{to } \frac{n}{2} \text{ terms})}{1.2.\;3 \;\text{-}\;\text{-}\;\text{-}\; \frac{n}{2}.\,2^{n}} - \frac{1}{2^{n-1}}$$

Whence the value of $PO^2 \times PO'^2 \times$ &c. above found becomes

$$\overline{x^2+1}^{\,n} - nx^2.\,\overline{x^2+1}^{\,n-2} + \frac{n.\overline{n-3}}{1.2}x^4 \times \overline{x^2+1}^{\,n-4} \;\text{-}\; + 2x^n$$

or expanding theſe terms

$$\left\{\begin{array}{l}
\overline{x^2+1}^{\,n} = x^{2n} + nx^{2n-2} + \frac{n.\overline{n-1}}{1.2}x^{2n-} + \frac{n.\overline{n-1}.\overline{n-2}}{1.2.3}x^{2n-6} + \text{-} \\
\quad \text{-}\;\text{-}\; + nx^2 + 1 \\
-nx^2 \times \overline{x^2+1}^{\,n-2} = -nx^{2n-2} - n.\overline{n-2}\,x^{2n-4} - \frac{n.\overline{n-2}.\overline{n-3}}{1.2} \\
x^{2n-6} \;-\;\text{-}\;\text{-}\;\text{-}\;\text{-}\;\text{-}\; - nx^2 \\
+\frac{n.\overline{n-3}}{1.2}x^4.\overline{x^2+1}^{\,n-4} = \frac{n.\overline{n-3}}{1.2}x^{2n-4} + \frac{n.\overline{n-3}.\overline{n-4}}{1.2}x^{2n-6} + \text{&c.} \\
-\frac{n.\overline{n-4}.\overline{n-5}}{1.2.3}x^6.\,\overline{x^2+1}^{\,n-6} = \quad -\frac{n.\overline{n-4}.\overline{n-5}}{1.2.3}x^{2n-6} \;\text{&c.} \\
\quad \text{&c.} \qquad\qquad\qquad\qquad\qquad\qquad \text{&c.}\;*
\end{array}\right\}$$

HENCE

* Mr. Simpſon in his Eſſays, page 115, has arrived by a different proceſs at a ſimilar concluſion, and aſſerts without any demonſtration that the co-efficients deſtroy each other. This however is the only difficult ſtep in the whole propoſition.

HENCE collecting the co-efficients it readily appears by considering the general value of the sum of u products stated above, that the coefficient of the term x^{2n-2m} the same as the coeff. of the term x^{2m} $=$

$$\left.\begin{array}{l}
+\dfrac{n.\,\overline{n-1}.\,\overline{n-2}\ \ -\ -\ -\ \ \overline{n-m-1}}{1.\ 2.\ 3\ \ -\ -\ -\ -\ \ m} \\[2ex]
-\dfrac{n.\,\overline{n-2}.\,\overline{n-3}\ \ -\ -\ -\ \ \overline{n-m}}{1.\ 1.\ 2\ \ -\ -\ -\ -\ \ m-1} \\[2ex]
-\ -\ -\ -\ -\ - \\
-\ -\ -\ -\ -\ - \\[1ex]
\pm\dfrac{n.\,\overline{n-m}.\,\overline{n-m+1}\ \ -\ -\ \ \overline{n-2m}}{1.\ 2.\ 3\ \ -\ -\ \ \overline{m-1}\times 1.\ 2} \\[2ex]
\pm\dfrac{n.\,\overline{n-m+1}.\,\overline{n-m+2}\ \ -\ \ \overline{n-2m+1}}{1.\ 2\ 3\ \ -\ -\ -\ \ m}
\end{array}\right\}$$

or reducing these fractions to a common denominator the numerator becomes $n\ \times$ into the expression in the Lemma which therefore $=$ o hence

$PO^2 \times PO'^2 \times$ &c. $= x^{2n} - 2x^n + 1$ or $PO \times PO' \times$ &c. $= x^n \backsim 1$. Q. E. D.

2. BECAUSE $PO \times PS \times PO' \times PS' \times$ &c. $= x^{2n} \backsim 1$ and $PO \times PO' \times$ &c. $= x^n \backsim 1$ $\therefore$ $PS \times PS' \times$ &c. $= x^n + 1$. Q. E. D.